Nigerian Journal of Management Studies
Vol. 18, No. 2, 2018, 45-55

Application of Control Theory to Water Quality Management

By

[1]Abiola, B. & [2]Ibiwoye, A.
babiola@unilag.edu.ng & aibiwoye @unilag.edu.ng
Department of Actuarial Science and Insurance,
University of Lagos, Nigeria.

ABSTRACT: *This paper considered the concentration of Biochemical Oxygen Demand (B.O.D) and Dissolved Oxygen as the two important indices of water quality management. The effect of uncertainties that could affect the rate of change of B.O.D and D.O. respectively were also considered. We used the model proposed by Streeter and Phelps enunciated in [Chen & Lee 1987], and developed an algorithm according to [Abiola& Ibiwoye 2012], in order to propose control signals that could move the D.O. level to some prescribed desired level.*

Keywords: Biochemical Oxygen Demand, Dissolved Oxygen, Control Signals, Uncertainties

1.0 Introduction

Water quality management has assumed a global recognition in recent time. In a Key note address delivered by Edwin D. Ongley at a Regional Workshop in Bangkok, Thailand in 1999, water quality management was identified as one of the major concern in Asia because water quality degradation is so severe in Asian countries, and it has been placing serious problem on economic growth.
There are wide spread evidence of destructive influences in near-shore and off-shore marine environment in Asia. According to Ongley (1998) coral destruction and contamination of marine life with serious implication for human health and dysfunctional behaviour of marine life are very evident in Asian countries. Countries such as China, India and Pakistan are suffering serious water deficit because of poor management of water quality.
In Nigeria issue of water pollution is evidently clear such that the Nigerian government had in 1988 need to set up the Federal Environmental Protection Agency (FEPA). Ready examples of pollution spots in Nigeria are the Lagoon front of the University of Lagos is being polluted regularly from pollutants from Ilaje and Bariga and the Niger delta. Pollution of water occurs frequently at the lower Niger Delta of Nigeria, as a result of petroleum spillage due to oil exploration activities. Artificially induced in-stream aeration method could have being the best method to restore polluted water to drinkable level in this region; however this is non-existence due to some environment factors.
The best considered option available is to breed micro-organism that consumes these pollutants. To implement this method on a large scale represents significant financial investment which require that our system must be designed optimally.
In 1999 the World Water Council presented the result of its visioning exercise, which is a vision to deal with water crisis, to the council of ministers in Netherland. In the report it seemed that by 2025, the quality of water will be restored to reasonable level, both for human and ecosystem purposes. The following anecdotal were made by some informed professionals.

- A world Bank study of the 1980s was cited as indicating that doubling of the Thai economy would increase water by a factor of $\times 10$
  - A similar study for India was cited for the period 1975-1995, based on a very conservative American Model, indicating a doubling of the economy would increase pollution by at least a factor of $\times 4$.
  - To achieve effective pollution control in India, the country would have spent some US$40 thousand million, on pollution which is vastly more than is spent in the entire water sector.

Nigerian Journal of Management Studies
Vol. 18, No. 2, 2018, 45-55

Additionally, the United Nations (1997) cites a UNIDO report that indicates that pollution in rapidly industrializing country of Asia, under a scenario of no change to water quality management policies will result in further pollution by a factor of up to $\times 8.$ clearly this is not sustainable. Therefore the need to apply management by exception techniques, in other words, management control becomes apposite when there is a deviation from expected result at every reach of the river.

2.0 Literature Review.
Two important indices of water quality in a river are Biochemical Oxygen Demand (BOD) and Dissolved Oxygen (DO). In the process of decomposing organic waste, micro- organisms consume oxygen dissolved in water. The larger the population of these micro-organisms, the greater is the demand for dissolved oxygen. If the level of (DO) in the stream is below a critical value, fishes and other aquatic animals, as well as many green plants are endangered.

(i) Enhance in-plant abatement
(ii) Flow augmentation from a reservoir system, and
(iii) Artificially induced in-stream aeration.

The last method represents a logical way to restore oxygen in simulated ecological manner, and according to Lee & Leitmann (1985), it is by far the cheapest. The method to reduce pollution for contaminated aquifer, resulting from petroleum spillage in the Niger Delta area of Nigeria was considered by Abiola (2009). In this paper we consider the case of reduction in river pollution and we shall make use of Streeter and Phelps's model.
Reveille and Ellis (1964) had observed that the issues inherent in water quality management constitute classic forms in public sector problems. It is not surprising, then, that much attention has been given to modeling the management of water quality over time. This attention is widespread. For instance, Albert (1988) listed the governmental responses to the pollution in Delaware that have resulted in the vastly improved water quality in that region. Similarly Daud (2018) made a clarion call for improvement in water quality in rivers that had served as the main source of water for consumption in Malaysia.
What all the attention seems to reveal is that the water quality control management problem is multidimensional in nature as there are physical, chemical as well as biological properties to contend with. From the comprehensive survey conducted by Jha, et. al (2015) it would appear that chemical properties based problems comprising Dissolved Oxygen (DO), pH, Biochemical Oxygen Demand (BOD), Nutrients, heavy metals and pesticides are more amenable to analytic techniques.
The basic model in this regard is the Streeter-Phelps partial differential equation developed in 1925 for determining the relationship between dissolved oxygen concentration and biological oxygen demand (Streeter and Phelps, 1925). A major modification of this model was carried out by Camp (1963) through the introduction of four additional parameters. A few years after, an interesting derivation from the Street-Phelps model was developed by Loucks et al (1967) using two linear programming models for determining the amount of waste water treatment required to achieve a minimum cost DO standards within a river basin. This was followed by another analytic model by Goodman and Tucker (1971) designed to simulate the variation of water quality with distance and time for a stream.

3.0 Method

Let x_1 and x_2 denote the concentration of BOD and OD respectively. According to Chen & Lee (1987) the following equations hold:

Nigerian Journal of Management Studies
Vol. 18, No. 2, 2018, 45-55

$$\begin{cases} \dfrac{\partial(\alpha x_1)}{\partial t} + \dfrac{\partial(\alpha v x_1)}{\partial s} = -k_1 \alpha x_1 + \alpha r_1(s,t) + h_1(t) \\ \dfrac{\partial(\alpha x_2)}{\partial t} + \dfrac{\partial(\alpha v x_2}{\partial s} = k_2 x_2 + k_3 \alpha (x_{2s} - x_2) + \alpha u(s,t) + h_2(t) \end{cases}$$

. . . (1)

Where

$\alpha =$ Cross section area.

$v =$ Velocity of the river

$s =$ Distance along the river.

$t =$ Time

$u =$ Artificial aeration rate (which is the control)

r_1 = B.O.D loading rate of all effluent discharge sources located along the river.

$h_1(t) =$ Uncertainties that affect B.O.D rate of change e.g. run-off and scour.

$h_2(t) =$ Uncertainty that affect D.O. rate of change e.g. photosynthesis

k_1, k_2, k_3 are positive constant.

The boundary conditions are

$$\begin{cases} x_1(s,0), x_2(s,0), 0 \le s \le S \\ x_1(0,t), x_2(0,t), 0 \le t \le T. \end{cases} \quad \text{. . . (2)}$$

The objective is to choose a control $u(s,t)$ such that the dissolved oxygen (D.O) level $x_2(t)$ is increased towards some desired level $x_{2D}(s,t)$.

The problem can be stated appropriately as follows

3.0 Problem (1):

Minimize $J(u) = \iint_\Omega \{(x_2(s,t) - x_{2D}(s,t))^2 q(s,t) + u^2(s,t)P(s,t)\} dt + \int_0^T \{(x_2(s,t) - x_{2D}(s,t))^2 f(t)\} dt$

Where $\Omega = (0,S) \times (0,T)$

Subject to the constraint:

Nigerian Journal of Management Studies
Vol. 18, No. 2, 2018, 45-55

$$\begin{cases} \dfrac{\partial(\alpha x_1)}{\partial t} + \dfrac{\partial(\alpha v x_1)}{\partial s} = -k_1 \alpha x_1 + \alpha r_1(s,t) + h_1(t) \\ \dfrac{\partial(\alpha x_2) +}{\partial t} \dfrac{\partial(\alpha v x_2}{\partial s} = k_2 x_2 + k_3 \alpha (x_{2s} - x_2) + \alpha u(s,t) + h_2(t) \end{cases}$$

. . . (3)

4.0 Methodology:

In Abiola & Ibiwoye (2012) a new algorithm to handle a class of problem defined below was developed:

Problem (2): $MinJ(u) = \int_0^T \{x(t)^T Qx(t) + u(t)^T Ru(t)\}dt$

Subject to the following constraint satisfaction:

$$x'(t) = Ax(t) + Bu(t) + Cv(t)$$

and

$x(0) = x^0$ given, $t \in [0,T]$ . . . (4)

$where, x(t) \in X \subset IR,^n$ is the state vector, $u(t) \in U \subset IR^m$, is the control vector and $v(t) \in V \subset IR^m$ is the uncertain vector, $U \cap V = \phi$, and T is the final time. If we set, see Abiola (2009)

$$J(u) =< u, (LQ^T L + R) + (m^0)^2 - (2m^0)(KQ^T L), u > +2 < z_1, Q^T Lu > -2m^0 <, K^T Qz >$$

. . . (5)

$$\nabla H(u) = 2\left\{u(t)R + BP^T \left[\Phi(t_0,t_0)x_0 + \int_0^t \Phi(t,\sigma)Bu(\sigma)d\sigma - \overline{m}\int_0^t \Phi(t.\sigma)u(\sigma)d\sigma\right]\right\}$$

. . . (6)

Where, $\nabla H(u)$ is the gradient of the Hamiltonian of the problem? With expressions (5) and (6), we can now recall the algorithm:

Now define $J(u)$ by equation (5) and $\nabla H(u)$ by (6) then

Step 1: choose u_0 arbitrarily and compute $J(u_0), \nabla H(u_0)$ and set

$\nabla H(u_k) = g_k, also, g_k = -F_k, k = 0$

Step2: If $\|g_k\| \le \varepsilon$, for a predetermined ε, stop else set

Nigerian Journal of Management Studies
Vol. 18, No. 2, 2018, 45-55

Step3: $u_{k+1} = u_k + \alpha_k W_k F_k(u_k), where$

$$\alpha_k = \frac{< g_k, g_k >}{< F_k, \bar{A} F_k >}$$

$$\bar{A} = \left\{ (LQ^T L + R) + ((m^0)^2 - 2m^0)(KQ^T L) \right\}$$

$$W_k = \left| \frac{P^{-1}}{\left[\int_0^T \left\{ \nabla H(u_k) P^{-1} \nabla H(u_k) \right\} d\tau \right]^{\frac{1}{2}}} \right|$$

To avoid the possibility of W_k being negative P is a positive definite matrix calculated from

$$PA + A^T P + Q = 0$$

Updating of the sequences is by gradient technique, which is defined as follows:

$$g_{k+1=} g_k + \alpha_k \bar{A} F_k$$

$$F_{k+1} = -g_{k+1} + \beta_k F_k$$

$$\beta_k = \frac{< g_{k+1}, g_{k+1} >}{< g_k, g_k >}$$

Step4: Set $k = k+1$, and go to step 1

To be able to apply the algorithm defined above there is need to convert problem (1) to the form of problem (2) and explicitly define the various operators concerned. This is what should set to accomplish in the subsequent section. There is need to reduce the Partial Differential Equation (P.D.E) defined in [1] to Ordinary Differential Equation (O.D.E.). This is done along the characteristic direction which satisfies:

$$\frac{ds}{dt} = v(s,t) \Leftrightarrow s = w(\sigma, t), t = W(\sigma, t) \qquad ...(7)$$

Let

$$x_2^* = x_2 - x_{2D}, h(t) = h_2(t) - h_1(t)$$

Define Transforming (s, t) into (σ, t) space we obtain the ordinary differential equation equivalent of (1) defined as

Nigerian Journal of Management Studies
Vol. 18, No. 2, 2018, 45-55

$$\begin{cases} x_1'(\sigma,t) = a_{11}(\sigma,t)x_1(\sigma,t) + r_1(\sigma,t) \\ x_2'(\sigma,t) = a_{21}(\sigma,t)x_1(\sigma,t) + a_{22}(\sigma,t)x_2(\sigma,t) + r_2(\sigma,t) + u(\sigma,t) \end{cases}$$

. . . (9)

The cost functional becomes:

$$J(u) = \int_{\sigma_0}^{\sigma f} J_\sigma(u)d\sigma.$$

Where

$$J_\sigma(u) = \int_{t_0(\sigma)}^{t_f(\sigma)} \{x_2^2(\sigma,t)q*(\sigma,t) + u^2(\sigma,t)P*(\sigma,t)\}dt + x_2^2(\sigma,t)f*(t_f(\sigma)) \quad ...(11)$$

Let <,> be the inner product, then we put (9), (10) and (11) into compact form to get:

Problem (3):

$$\begin{cases} MinJ(u) = \int_{\sigma_0}^{\sigma f} J_\sigma(u)d\sigma \\ J_\sigma(u) = \int_{t_0(\sigma)}^{t_f(\sigma)} \{< x, Qx > + < u, Ru >\}dt + < x, Fx > \end{cases}$$

Subject to,

$$x'(t) = A(\sigma,t)x(\sigma,t) + Bu(\sigma,t) + r_2(\sigma,t) \quad ...(12)$$

Where,

$$x = (x_1, x_2)$$

$$\begin{cases} A = \begin{bmatrix} a_{11}(\sigma,t) & 0 \\ a_{21}(\sigma,t) & a_{22}(\sigma,t) \end{bmatrix} \\ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ Q = q*(\sigma,t), R = P*(\sigma,t), F = f*(\sigma,t). \end{cases}$$

. . . (13)

Nigerian Journal of Management Studies
Vol. 18, No. 2, 2018, 45-55

From expression in (12) and (13) we can now see that we have transformed problem (1) into the form of problem (2).
In what now follows, we define the Hamiltonian of problem (3) as:

$$H(x,u,r,\lambda) = x(\sigma,t)^T Qx(\sigma,t) + u(\sigma,t)^T Ru(\sigma,t) + \lambda^T (A(\sigma,t)x(\sigma,t) + Bu(\sigma,t) + r(\sigma,t) \quad \ldots (14)$$

Consider (14) and find the Frechet derivatives with respect to t, as follows:

$$H(x,u+h,r,\lambda) = x^T Q(\sigma,t)x + (u+h)^T R(u+h) + \lambda(A(\sigma,t)x(t) + B(u+h) + r(t) \quad \ldots (15)$$

$$H(x,u+h,r,\lambda) - H(x,u,r,\lambda) = 2u(\sigma,t)Rh + \lambda^T Bh$$

$$\lim_{h\to 0} \frac{H(x,u+h,r,\lambda) - H(x,u,r,\lambda)}{\|h\|} = 2u(\sigma,t)^T R + \lambda^T B \quad \ldots (16)$$

Choose $\lambda(t) = 2Kx(\sigma,t)$ then, (16) is expressible as:

$$2\{u(\sigma,t)^T R + BK^T (x(\sigma,t))\} \quad \ldots (17)$$

We need to determine $x(\sigma,t)$ from

$$x'(\sigma,t) = A(\sigma,t)x(\sigma,t) + Bu(\sigma,t) + r(\sigma,t) \quad \ldots (18)$$

We now consider

$$x'(\sigma,t) = A(\sigma,t)x(\sigma,t) \quad \ldots (19)$$

Let $\Theta(t_0,t)$ be the fundamental matrix for (19) then:

$$x(\sigma,t) = \Theta(t_0,t)x_0 + \int_0^t \Theta(t_0,t)\{Bu(\sigma,t) + r(\sigma,t)\}dt \quad \ldots (20)$$

By virtue of (20) we can express (17) as:

$$\nabla H = 2\left\{u(\sigma,t)^T R + BK^T \left[\Theta(t_0,t)x_0 + \int_0^t \Theta(t_{0,}t)\{Bu(\sigma,t) + r(\sigma,t)\}dt\right]\right\} \quad \ldots (21)$$

If we are to use the algorithm earlier proposed by Abiola and Ibiwoye we need an explicit determination of the operator $\overline{A}$ defined in step111 of the algorithm earlier proposed.

Nigerian Journal of Management Studies
Vol. 18, No. 2, 2018, 45-55

In what now follows, we shall determine the operator $\bar{A}$

4.0 Determination of Operator $\bar{A}$

Consider:

Min $J(u) = \int_{\sigma_0}^{\sigma f} J_\sigma(u) d\sigma$

Where

$$J_\sigma(u) = \int_{t_0(\sigma)}^{t_f(\sigma)} \{ < x, Qx > + < u, Ru > \} dt + < x, Fx > \quad \ldots (22)$$

Subject to

$$x'(\sigma, t) = A(\sigma, t) x(\sigma, t) + Bu(\sigma, t) + r_2(\sigma, t) \quad \ldots (23)$$

Let $B[IR^n, IR^m]$ denote the set of bounded linear map between IR^n and IR^m, and let

$L : B[IR^n, IR^m] \to B[IR^n, IR^m]$ Be such that:

$$L(u) = \int_0^t \Phi(t_0, t) Bu(\sigma) d\sigma \quad \ldots (24)$$

Also, define:

$$Kr_2(\sigma, t) = \int_0^t \Phi(t_0, t) r_2(\sigma, t) d\sigma \quad \ldots (25)$$

Define the inner product on IR^n as follows:

For $w_1, w_2 \in IR^n$

$$< w_1, w_2 > = \int_0^T w_1(t) w_2(t) dt \quad \ldots (26)$$

Let $\Phi(t_0(\sigma), t(\sigma)) x(\sigma, 0) = \gamma_1 \quad \ldots (27)$

It is an easy matter to observe that:

$$x(\sigma, t) = \gamma_1 + Lu + Kr_2(\sigma, t) \quad \ldots (28)$$

We now express:

Nigerian Journal of Management Studies
Vol. 18, No. 2, 2018, 45-55

$$J(u) = \int_{\sigma_0}^{\sigma f} \left\{ \int_{t_0(\sigma)}^{t_f(\sigma)} \{ < \gamma_1 + Lu + Kr_2(\sigma,t), Q(\gamma_1 + Lu + Kr_2(\sigma,t) > + < u, Ru > \} dt \right\} d\sigma + \\ < \gamma_1 + Lu + Kr_2(\sigma,t), F(\gamma_1 + Lu + Kr_2(\sigma,t) > \quad \ldots (29)$$

On expansion we arrive at the following expression:

$$J(u) = \int_{\sigma_0}^{\sigma f} \left\{ \int_{t_0(\sigma)}^{t_f(\sigma)} \{ < u, (LQ^T L + R)u > + < \gamma_1, KQ^T Lu > + 2 < u, LQ^T Kr_2 > + 2 < r_2, K^T Q\gamma_1 > \\ < r_2, KQ^T Kr_2 > \} dt \right\} d\sigma + < \gamma_1 + Lu + Kr_2(\sigma,t) F(\gamma_1 + Lu + Kr_2(\sigma,t)) >$$

$$\ldots (30)$$

We expand carefully to arrive at the following expression:

$$J(u) = \int_{\sigma_0}^{\sigma f} \left\{ \int_{t_0(\sigma)}^{t_f(\sigma)} < u, (LQ^T L + R)u > + < r_2, KQ^T Lu > + 2 < u, LQ^T Kr_2 > + 2 < r_2, Q^T Lu > + 2 < r_2, K^T Q\gamma_1 > \\ < r_2, KQ^T Kr_2 > \} dt \right\} d\sigma$$

$$\ldots (31)$$

A quick check on the right-hand –side of (31) shows that the expression
$< u, (LQ^T L + R)u >, and, < \gamma_1, Lu + Kr_2(\sigma,t) F(\gamma, Lu + Kr_2(\sigma,t) >$

play a dominant role in the minimization process. Therefore we set our required operator

$$\overline{A} = < u, (LQ^T L + R)u > + < \gamma_1, Lu + Kr_2(\sigma,t) F(\gamma_1, Lu + Kr_2(\sigma,t) > \quad \ldots (32)$$

On application of (32) in the procedure define in the solution methodology we can now redefine our new algorithm as proposed below:

5.0 Required Algorithm

Step 1: choose u_0 arbitrarily and compute $J(u_0), \nabla H(u_0)$ and set

$\nabla H(u_k) = g_k, also, g_k = -F_k, k = 0$

Step2: If $\|g_k\| \le \varepsilon$, for a predetermined ε, stop else set

Nigerian Journal of Management Studies
Vol. 18, No. 2, 2018, 45-55

Step3: $u_{k+1} = u_k + \alpha_k W_k F_k(u_k), where$

$$\alpha_k = \frac{< g_k, g_k >}{< F_k, \bar{A}F_k >}$$

$$\bar{A} = \left\{(LQ^T L + R) + ((m^0)^2 - 2m^0)(KQ^T L)\right\}$$

$$W_k = \left| \frac{P^{-1}}{\left[\int_0^T \left\{\nabla H(u_k) P^{-1} \nabla H(u_k)\right\} d\tau\right]^{\frac{1}{2}}} \right|$$

To avoid the possibility of W_k being negative P is a positive definite matrix calculated P from

$$PA + A^T P + Q = 0$$

… (33)

Updating of the sequences is by gradient technique, which is defined as follows:

$$g_{k+1=} g_k + \alpha_k \bar{A} F_k$$

$$F_{k+1} = -g_{k+1} + \beta_k F_k$$

$$\beta_k = \frac{< g_{k+1}, g_{k+1} >}{< g_k, g_k >}$$

Step4: Set $k = k + 1$, and go to step 1

The operator $\bar{A}$ is defined in equation (32).

6.0 Conclusion: It is important to emphasis here that the mathematical determination of the control programme is very important. This is because it would be very easy to determine which management control would be appropriate to apply at every reach of the river in order to reduce the level of pollutants that will reduce the rate of contamination of the river to a manageable level that will support aquatic lives. Therefore, with the explicit determination of operator $\bar{A}$, it becomes an easy matter for us to determine the control signals that would help move D.O. level to some prescribed desired level. This is left as a subject of further research.

7.0 References

Abiola, B. (2009). Stable control for contaminated aquifer in an uncertain environment: Science World Journal Vol. 4, No. 3

Nigerian Journal of Management Studies
Vol. 18, No. 2, 2018, 45-55

Abiola, B. & Ibiwoye, A. (2012). On Numerical Algorithm for Uncertain System: Science World Journal7

Albert, R. C. (1988). The Historical Context of Water Quality Management for Delaware Estuary, Estuaries Vol. 11 No.2, pp. 99-107

Chen, Y. H. & Lee, C. S. (1987). On the control of an Uncertain Water Quality System: Optimal Control, Application and Methods Vol. 8, pp. 279-298

Camp, T. R. (1963). First Expanded BOD – DO model. In Basic Water Quality Models: (IHP – V Project 8) (edited by D.G. Jolankai), 27-29 UNESCO Publication

Daud, H. (2018). Legislative Approach to Water Quality Management in Malaysia, Success and Challenges www.wepa.db.net/pdf/o810forum/paper34,pdf retrieved 5/28/2018

Goodman. A. S. & Tucker, R. J. (1971). Time varying mathematical model for water quality. Water Research, Vol. 5, Iss. 5, pp 227-241

Jha, et. al. (2015). Multiobjective design and optimization of hard magnetic alloy free of rare-earth. Contributed paper from material science and technology. Oct. 04-08-2015, Greater Columbus Convention Centre, Ohio, USA

Lee, C. S. & Leitmann, G. (1987). Deterministic Single Reach River. A discrete Model: Proceeding of SICE Hiroshima, Japan. Vol. 3, No. 5.

Loucks, D. P., Revelle, C. S. & Walter, R. L. (1967). Linear Programming models for Water Pollution Control. Management Science. Vol.14, Iss. 4. pp. 166-181

Revelle, C. & Ellis, J. H. (1994). Model for Water and Air Quality Management in Handbooks in Operation Research and Management Science, S.M. Polock, M.H. Rothkolf and A.Barnett Eds. Vol. 6, pp. 365-401

Ongley, E.D (1998). Modernization of Water Quality Management in Developing Countries; Issue of Relevancy and Cost Efficiency. Water Quality International Sept/Oct 1998 pp. 37-42

Steeter, H. W. & Phelps, E. B. (1925). A Study of the pollution and natural purification of the Ohio River. Factors concerned in the phenomena of oxidation and recreation, Public Health Bulletin no. 146, Reprinted by U.S. Department of Health, Education and Welfare. Public Health Service, 1958, ISBN B001BP4GZI